

# Four-potentials and Maxwell Field Tensors from $SL(2, C)$ Spinors as Infinite-Momentum/Zero-Mass Limits of their Massive Counterparts

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Four  $SL(2, C)$  spinors are considered within the framework of Wigner's little groups which dictate internal space-time symmetries of relativistic particles. It is indicated that the little group for a massive particle at rest is  $O(3)$ , while it is  $O(3)$ -like for moving massive particles. The little group becomes like  $E(2)$  in the infinite-momentum/zero-mass limit. Spin- $\frac{1}{2}$  particles are studied in detail, and the origin of the gauge degrees of freedom for massless particles is clarified. There are sixteen different combinations of direct products of two  $SL(2, C)$  spinors for spin-1 and spin-0 particles. The state vectors for the  $O(3)$  and  $O(3)$ -like little groups are constructed. It is shown that in the infinite-momentum/zero-mass limit, these state vectors become scalars, four-potentials and the Maxwell field tensor. It is revealed that the Maxwell field tensor so obtained corresponds to some of the state vectors constructed by Weinberg in 1964 [S. Weinberg, Phys. Rev. **B135** 1049 (1964)].

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## I. INTRODUCTION

One of the beauties of Einstein's special relativity is the unified description of the energy-momentum relation for massive and massless particles through  $E = [(cP)^2 + (Mc^2)^2]^{1/2}$ . In 1939 Wigner observed that in addition to mass, energy and momentum, relativistic particles have internal space-time degrees of freedom [1]. For this purpose, he studied the subgroups of the Poincaré group whose transformations leave invariant the four-momentum of a given free particle. The Poincaré group is the group of inhomogeneous Lorentz transformations, namely Lorentz transformations followed by space-time translations. The two Casimir operators of this group correspond to the  $(\text{mass})^2$  and  $(\text{spin})^2$  of a given particle. The maximal subgroup of the Lorentz group which leaves the four-momentum invariant is called the little group. Wigner showed that the internal space-time symmetries of massive and massless particles are dictated by the  $O(3)$ -like and  $E(2)$ -like little groups respectively. Here, the word "like" is used to indicate that the Lie algebra is the same. The  $O(3)$ -like little group has the same Lie algebra as that of  $O(3)$ .

Electron spins are manifestations of internal space-time structure of elementary particles. They are described by the Dirac equation, which constitutes a non-unitary finite-dimensional representation of Wigner's little group. Indeed, the Dirac equation is a four-by-four representation of the  $SL(2, C)$  group. In this paper, we shall examine systematically the  $SL(2, C)$  spinors. While there are two orthogonal spinors in the Pauli representation of  $SU(2)$ , there are four different spinors in the  $SL(2, C)$  regime [2,3]. From these, we shall construct representations of the  $O(3)$ -like little group,  $E(2)$ -like little group, and the contraction of the  $O(3)$ -like little group to the  $E(2)$ -like little group in the infinite-momentum/zero-mass limit.

It was shown by İnönü and Wigner that the rotation group  $O(3)$  can be contracted to  $E(2)$  [4]. An important development along this line of research is the application of group contractions to the unification of the two different little groups for massive and massless particles. The  $E(2)$ -like little group for massless particles is obtained from the  $O(3)$ -like little group for massive particles in the infinite-momentum/zero-mass limit [5].

Like the three-dimensional rotation group,  $E(2)$  is a three-parameter group. It contains two translational degrees of freedom in addition to the rotation. The physics associated with the translational-like degrees of freedom for the case of the  $E(2)$ -like little group has been shown by various authors to be the gauge degrees of freedom for massless particles [7,6]. Indeed, these gauge degrees of freedom emanate from the contraction of the transverse components of the rotation generators during the contraction process of the  $O(3)$ -like little group to the  $E(2)$ -like little group [8]. This contraction process has been studied in detail by Kim and Wigner in terms of cylindrical geometry [9].

As far as  $SL(2, C)$  is concerned, it has been possible to interpret the Dirac spinors within the framework of Wigner's representation theory [3]. Is it then possible to find a place for Maxwell's field tensors in the same theory? In 1964, Weinberg formulated the process of constructing all representations for massless fields starting from the  $SL(2, C)$  spinors [10]. However, he uses only the spinors which are invariant under the translation-like transformations of the

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$E(2)$ -like little group. Since these transformations are gauge transformations, Weinberg's construction contains the gauge-invariant Maxwell field tensor. Weinberg's construction does not include gauge-dependent four-potentials.

It has earlier been discussed in the literature that the electromagnetic four-potential can be obtained from the group-contraction procedure [9]. But the contraction procedure for the field tensor has not been discussed. We address this problem in the present paper. In order to have a coherent presentation of this procedure, we construct both the four-vector and the second-rank tensor from the four  $SL(2, C)$  spinors for massive particles at rest. We then boost those spinors resulting in boosting the four-vector and the second-rank tensor. When the boost parameter approaches infinity, they become the electromagnetic four-potential and the Maxwell field tensor. In this way we establish a covariant picture of the little group and its representation space for both massive and massless particles.

In Sec. II, we review the little groups and emphasize that they are Lorentz-covariant entities. The  $O(3)$ -like little group is not only valid in the rest frame of the massive particle, but also in all other Lorentz frames. If the particle is not at rest, the little group is not  $O(3)$ , but is an  $O(3)$ -like group which is a covariant entity. In Sec. III, the spin 0 and spin 1 representations of the little group are constructed from the  $S(2, C)$  spinors. Since there are four  $SL(2, C)$  spinors for a spin- $\frac{1}{2}$  particle, there are four different types of combinations of the usual  $SU(2)$  construction. In Sec. IV, four-vectors and second rank tensors are constructed from the  $SL(2, C)$  spinors. They are covariant entities which can be boosted from the rest frame to an infinite-momentum frame. Within the framework of the Inönü-Wigner contraction, they become the four-potentials and the Maxwell field tensor. It is noted that the field tensor corresponds to some of the state vectors constructed by Weinberg.

## II. LITTLE GROUPS OF THE POINCARÉ GROUP

The group of Lorentz transformations is generated by three rotation generators  $J_i$  and three boost generators  $K_i$ . They satisfy the commutation relations of the  $SL(2, C)$  Lie algebra :

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k. \quad (2.1)$$

For a massive point particle there is a Lorentz frame in which the particle is at rest. In this frame, the little group is clearly the three-dimensional rotation group  $O(3)$ . The four-momentum is not affected by this rotation, but the spin variable changes its direction. Hence, the little group of the moving massive particle can be obtained by boosting along the direction of the momentum. Without loss of generality, if the particle is boosted in the  $z$  direction, the generators of the little group can be obtained by

$$J'_i = B(\eta)J_iB(\eta)^{-1}, \quad (2.2)$$

where  $B(\eta) = \exp(-i\eta K_3)$ . Since  $J_3$  commutes with  $K_3$ , it remains invariant, while  $J_1$  and  $J_2$  assume the form

$$J'_1 = \cosh \eta J_1 + \sinh \eta K_2, \quad J'_2 = \cosh \eta J_2 - \sinh \eta K_1, \quad (2.3)$$

which satisfy the Lie algebra of  $O(3)$ . Thus, a moving massive particle still has its  $O(3)$ -like little group.

However, for massless particles there are no Lorentz frames in which the particle is at rest. The approach is to consider the limiting case in which the mass of the particle becomes vanishingly small yielding the boost parameter to become infinite. After renormalizing the generators  $J'_1$  and  $J'_2$  as

$$N_1 = -(\cosh \eta)^{-1}J'_2, \quad N_2 = (\cosh \eta)^{-1}J'_1 \quad (2.4)$$

in the infinite- $\eta$  limit they reduce to

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (2.5)$$

The operators  $N_1$ ,  $N_2$  and  $J_3$  satisfy the commutation relations

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0, \quad (2.6)$$

where  $J_3$  is like the rotation generator, while  $N_1$  and  $N_2$  are like translation generators in the two-dimensional Euclidean plane. Hence, they are the generators of the  $E(2)$ -like little group for massless particles.

The traditional approach to the little groups has been to emphasize the difference between those for massive and massless particles. In this paper, we would like to emphasize that the little group is a covariant entity and remains  $O(3)$ -like in all Lorentz frames. It becomes  $E(2)$ -like in the infinite-momentum/zero-mass limit as in the case of the Inönü-Wigner contraction in which  $O(3)$  becomes  $E(2)$ .

### III. $SL(2, C)$ SPINORS

The commutation relations of the three dimensional rotation group is contained in the  $SL(2, C)$  algebra introduced in (2.1) as

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \quad (3.1)$$

The two-by-two representation of this group is called  $SU(2)$ , and the generators are given in terms of the Pauli spin matrices  $\sigma_i$ . We observe that the set of commutation relations for the  $SL(2, C)$  algebra is not invariant under the sign change of the rotation generators, but remains invariant under the sign change of the boost generators. Thus, the first and the second solution of this set of commutation relation consists of

$$J_i = \frac{1}{2}\sigma_i, \quad K_i = \frac{i}{2}\sigma_i, \quad (3.2)$$

and

$$J_i = \frac{1}{2}\sigma_i, \quad \dot{K}_i = -\frac{i}{2}\sigma_i, \quad (3.3)$$

respectively. We call these two representations “undotted” and “dotted” representations respectively.

When we refer to the  $SU(2)$  subgroup of  $SL(2, C)$ , it is usually understood to be the subgroup in the undotted representation. The representation space consists of

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.4)$$

However, if we take into account the boost generators, there are two different sets of representation spaces. We use the notation  $\dot{\alpha}$  and  $\dot{\beta}$  for the spinors for the dotted representation, and they also take the form

$$\dot{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \dot{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.5)$$

However, these two sets of spinors have quite different Lorentz-boost properties. There are therefore four independent spinors. This is why the Dirac spinor has four components.

For simplicity, we shall consider rotations around and boosts along the  $z$  direction. The rotation matrix both in the dotted and undotted representation is

$$R(\theta) = \exp(-i\theta J_3) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}. \quad (3.6)$$

The boost matrix in the undotted representation is

$$B(\eta) = \exp(-i\eta K_3) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \quad (3.7)$$

while it becomes

$$\dot{B}(\eta) = \exp(-i\eta \dot{K}_3) = \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}, \quad (3.8)$$

for the dotted representation, where  $K_3$  and  $\dot{K}_3$  take the form  $(i/2)\sigma_3$  and  $(-i/2)\sigma_3$  respectively. Therefore we have

$$\begin{aligned} B(\eta)\alpha &= e^{\eta/2}\alpha, & B(\eta)\beta &= e^{-\eta/2}\beta, \\ \dot{B}(\eta)\dot{\alpha} &= e^{-\eta/2}\dot{\alpha}, & \dot{B}(\eta)\dot{\beta} &= e^{\eta/2}\dot{\beta}. \end{aligned} \quad (3.9)$$

Since the spinorial space is generated by four independent spinors, the little groups for these spinors are to be constructed accordingly. As was emphasized in Sec. II, the generators of the  $E(2)$ -like little group are obtained from the contraction procedure defined in (2.5). During this process,  $J_3$  remains unchanged, and  $N_1$  and  $N_2$  applicable to  $\alpha$  and  $\beta$  become

$$N_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.10)$$

while for  $\dot{\alpha}$  and  $\dot{\beta}$  one has to employ

$$\dot{N}_1 = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \quad \dot{N}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3.11)$$

Related to these generators the transformation matrices are obtained by  $D(u, v) = \exp(-iuN_1 - ivN_2)$  and  $\dot{D}(u, v) = \exp(-iu\dot{N}_1 - iv\dot{N}_2)$ , and explicitly written as

$$D(u, v) = \begin{pmatrix} 1 & u - iv \\ 0 & 1 \end{pmatrix}, \quad \dot{D}(u, v) = \begin{pmatrix} 1 & 0 \\ -u - iv & 1 \end{pmatrix}. \quad (3.12)$$

Two of the spinors are invariant under these transformations

$$D(u, v)\alpha = \alpha, \quad \dot{D}(u, v)\dot{\beta} = \dot{\beta} \quad (3.13)$$

while the other two are “gauge-dependent” in the sense that

$$D(u, v)\beta = \beta + (u - iv)\alpha, \quad \dot{D}(u, v)\dot{\alpha} = \dot{\alpha} - (u + iv)\dot{\beta}. \quad (3.14)$$

The “gauge-invariant” spinors of (3.13) appear as polarized neutrinos in the real world. Indeed, it has been revealed that the polararization of neutrinos is a consequence of gauge invariance [11]. There are no neutrinos which correspond to the gauge-dependent spinors  $\dot{\alpha}$  and  $\dot{\beta}$ , since they represent unobservable quantities. It has been shown that these gauge-dependent spinors are responsible for the gauge dependence of the electromagnetic four-vector [12]. This issue will further be elaborated in Sec. IV A.

Let us now construct spin-1 and spin-0 state vectors using two spinors. In the  $SU(2)$  and  $O(3)$  regime, we construct the spin-1 and spin-0 wave functions by making the following combinations:

$$\alpha\alpha, \quad \frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha), \quad \beta\beta, \quad \frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha). \quad (3.15)$$

This will be all, if we restrict ourselves to the undotted representation. If we restrict ourselves to the dotted representation we have

$$\dot{\alpha}\dot{\alpha}, \quad \frac{1}{\sqrt{2}}(\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\alpha}), \quad \dot{\beta}\dot{\beta}, \quad \frac{1}{\sqrt{2}}(\dot{\alpha}\dot{\beta} - \dot{\beta}\dot{\alpha}). \quad (3.16)$$

Here the story is the same for  $O(3)$ . It is also possible to make mixed combinations:

$$\begin{aligned} \alpha\dot{\alpha}, & \quad \frac{1}{\sqrt{2}}(\alpha\dot{\beta} + \dot{\beta}\alpha), & \beta\dot{\beta}, & \quad \frac{1}{\sqrt{2}}(\alpha\dot{\beta} - \dot{\beta}\alpha), \\ \dot{\alpha}\alpha, & \quad \frac{1}{\sqrt{2}}(\dot{\alpha}\beta + \beta\alpha), & \dot{\beta}\beta, & \quad \frac{1}{\sqrt{2}}(\dot{\alpha}\beta - \beta\alpha). \end{aligned} \quad (3.17)$$

All of the above four combinations have the same property under rotations. However, they become different if the Lorentz boosts are taken into account.

It is known that the combinations in (3.17) have the same transformation property as that of a four-vector under Lorentz transformations. It has also been mentioned that particular combinations in (3.15) and (3.16) behave like tensors [2,3]. In the next section, we prove that this is indeed so. Furthermore, we consider the infinite-momentum/zero-mass limits of these combinations. We shall show that the four-vector becomes like the four-potential, and the tensor becomes like the electromagnetic fields.

## IV. FOUR-VECTORS AND TENSORS FROM $SL(2, \mathbb{C})$ SPINORS

### A. Four-vectors

Let us consider the direct products of one dotted and one undotted spinor as  $\alpha\dot{\alpha}$ ,  $\alpha\dot{\beta}$ ,  $\dot{\alpha}\beta$ ,  $\beta\dot{\beta}$  and a four-vector  $V^\mu = (V_x, V_y, V_z, V_t)$  representing a massive particle. This vector can be expressed in the form of a two-by-two

Hermitian matrix by  $V = \sum \sigma_\mu V^\mu$  or equally well by  $\dot{V} = \sigma_0 V_t - \sum \sigma_i V^i$ , where  $\sigma_0$  is the identity matrix. As the notation suggests  $\dot{V}$  admits  $\dot{K}_i$  as the generators of boosts. The matrix  $V$  explicitly takes the form

$$V = \begin{pmatrix} V_t + V_z & V_x - iV_y \\ V_x + iV_y & V_t - V_z \end{pmatrix}, \quad (4.1)$$

and the transformation rules under rotations and boosts are explained in App. A. The transformation properties of the direct products of spinors are derivable from Sec. III.

TABLE I. Spinorial combinations for four-vectors

Spinorial combinations	Transform like	Under $R(\theta)$	Under $B(\eta)$
$-\alpha\dot{\alpha}$	$V_x - iV_y$	$e^{-i\theta}$	$e^0$
$\alpha\dot{\beta}$	$V_t + V_z$	$e^0$	$e^\eta$
$-\dot{\alpha}\beta$	$V_t - V_z$	$e^0$	$e^{-\eta}$
$\beta\dot{\beta}$	$V_x + iV_y$	$e^{i\theta}$	$e^0$

In view of Table I, we can make the following identifications between the components  $V^\mu$  and the direct products of spinors as:

$$\begin{aligned} V_x &\simeq \frac{1}{2}(\beta\dot{\beta} - \alpha\dot{\alpha}), & V_z &\simeq \frac{1}{2}(\alpha\dot{\beta} + \dot{\alpha}\beta), \\ V_y &\simeq \frac{i}{2}(\beta\dot{\beta} + \alpha\dot{\alpha}), & V_t &\simeq \frac{1}{2}(\alpha\dot{\beta} - \dot{\alpha}\beta). \end{aligned} \quad (4.2)$$

We observe that  $V_x$  and  $V_y$  are invariant quantities considering both rotations around and boosts along the  $z$ -axis. On the other hand the components  $V_z$  and  $V_t$  are squeezed, since they transform as

$$V'_z \simeq \frac{1}{2}(e^\eta \alpha\dot{\beta} + e^{-\eta} \dot{\alpha}\beta), \quad V'_t \simeq \frac{1}{2}(e^\eta \alpha\dot{\beta} - e^{-\eta} \dot{\alpha}\beta) \quad (4.3)$$

under boosts.

## B. Scalars and $2^{nd}$ -Rank Tensors

As it has been possible to construct four-vectors from the direct product of two spinors, it is also possible to construct scalars and antisymmetric  $2^{nd}$ -rank tensors. Antisymmetric combinations  $\frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha)$  and  $\frac{1}{\sqrt{2}}(\dot{\alpha}\dot{\beta} - \dot{\beta}\dot{\alpha})$  are invariant under rotations and boosts, and therefore have similar transformation properties with that of scalars.

A way to achieve an antisymmetric  $2^{nd}$ -rank tensor is to consider the direct products of two dotted and the direct products of two undotted spinors

$$(i) \ \alpha\alpha, \ \alpha\beta, \ \beta\beta \quad \text{and} \quad (ii) \ \dot{\alpha}\dot{\alpha}, \ \dot{\alpha}\dot{\beta}, \ \dot{\beta}\dot{\beta}.$$

Under the action of the Lorentz group, particular combinations of the two categories (i) and (ii) are transformed into combinations of each other. Consider an antisymmetric four-by-four matrix

$$T = \begin{pmatrix} 0 & -g_z & g_y & f_x \\ g_z & 0 & -g_x & f_y \\ -g_y & g_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix}. \quad (4.4)$$

It is well known that  $f = (f_x, f_y, f_z)$  and  $g = (g_x, g_y, g_z)$  transform like three-vectors under rotations. Transformation properties under boosts are also well known. Here we are interested in studying the transformation properties of the above tensor in terms of the  $SL(2, C)$  spinors. For this purpose, we should be able to write each element of the tensor  $T$  in terms of  $SL(2, C)$  spinors. The boost matrix for the tensor  $T$  is

$$B(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \eta & \sinh \eta \\ 0 & 0 & \sinh \eta & \cosh \eta \end{pmatrix}. \quad (4.5)$$

We note first that  $f_z$  and  $g_z$  are invariant under this boost. In addition, these quantities are invariant under the rotation around the  $z$ -axis, where the rotation matrix is

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.6)$$

The spinor combinations which satisfy these invariance requirements are

$$\frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha), \quad \frac{1}{\sqrt{2}}(\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\alpha}). \quad (4.7)$$

For this reason, we are allowed to identify them with  $f_z$  and  $g_z$  respectively. Let us introduce the subsequent notations:

$$\begin{aligned} \kappa_+ &= (f_x + ig_x) + i(f_y + ig_y), & \kappa_- &= (f_x + ig_x) - i(f_y + ig_y), \\ \lambda_+ &= (f_x - ig_x) + i(f_y - ig_y), & \lambda_- &= (f_x - ig_x) - i(f_y - ig_y). \end{aligned} \quad (4.8)$$

We observe that these quantities can be identified with

$$\kappa_- \simeq \alpha\alpha, \quad \kappa_+ \simeq \beta\beta, \quad \lambda_- \simeq \dot{\alpha}\dot{\alpha}, \quad \lambda_+ \simeq \dot{\beta}\dot{\beta}, \quad (4.9)$$

since they exhibit the same transformation properties which can directly be derivable from those of the  $SL(2, C)$  spinors given in Sec. III.

TABLE II. Spinorial combinations and  $2^{nd}$ -rank tensors

Spinorial combinations	Transform like	Under $R(\theta)$	Under $B(\eta)$
$\alpha\alpha$	$\kappa_-$	$e^{-i\theta}$	$e^\eta$
$\frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha)$	$f_z$	$e^0$	$e^0$
$\beta\beta$	$\kappa_+$	$e^{i\theta}$	$e^{-\eta}$
$\dot{\alpha}\dot{\alpha}$	$\lambda_-$	$e^{-i\theta}$	$e^{-\eta}$
$\frac{1}{\sqrt{2}}(\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\alpha})$	$g_z$	$e^0$	$e^0$
$\dot{\beta}\dot{\beta}$	$\lambda_+$	$e^{i\theta}$	$e^\eta$

In view of Table II,  $f$  and  $g$  can be written in terms of  $\kappa_\pm$  and  $\lambda_\pm$  as

$$\begin{pmatrix} f_x \\ f_y \\ g_x \\ g_y \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -i & i & -i & i \\ -i & -i & i & i \\ -1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \kappa_+ \\ \kappa_- \\ \lambda_+ \\ \lambda_- \end{pmatrix}, \quad (4.10)$$

or more explicitly as

$$\begin{aligned} f_x &\simeq \frac{1}{4}(\alpha\alpha + \dot{\beta}\dot{\beta} + \beta\beta + \dot{\alpha}\dot{\alpha}), & g_x &\simeq \frac{i}{4}(-\alpha\alpha + \dot{\beta}\dot{\beta} - \beta\beta + \dot{\alpha}\dot{\alpha}), \\ f_y &\simeq \frac{i}{4}(\alpha\alpha - \dot{\beta}\dot{\beta} - \beta\beta + \dot{\alpha}\dot{\alpha}), & g_y &\simeq \frac{1}{4}(\alpha\alpha + \dot{\beta}\dot{\beta} - \beta\beta - \dot{\alpha}\dot{\alpha}). \end{aligned} \quad (4.11)$$

Under the Lorentz boost along the  $z$ -direction, these quantities become

$$\begin{aligned} f'_x &\simeq \frac{1}{4}[e^\eta(\alpha\alpha + \dot{\beta}\dot{\beta}) + e^{-\eta}(\beta\beta + \dot{\alpha}\dot{\alpha})], & g'_x &\simeq \frac{i}{4}[e^\eta(\alpha\alpha + \dot{\beta}\dot{\beta}) + e^{-\eta}(-\beta\beta + \dot{\alpha}\dot{\alpha})], \\ f'_y &\simeq \frac{i}{4}[e^\eta(\alpha\alpha - \dot{\beta}\dot{\beta}) + e^{-\eta}(-\beta\beta + \dot{\alpha}\dot{\alpha})], & g'_y &\simeq \frac{1}{4}[e^\eta(\alpha\alpha + \dot{\beta}\dot{\beta}) + e^{-\eta}(-\beta\beta - \dot{\alpha}\dot{\alpha})]. \end{aligned} \quad (4.12)$$

## V. MASSLESS PARTICLES

As was shown by Inönü and Wigner [4], the group  $O(3)$  can be contracted to  $E(2)$ , and the contraction procedure is well defined. It is essentially a large-radius approximation of a finite area on a spherical surface. Likewise, contraction of the  $O(3)$ -like little group to the  $E(2)$ -like little group can be defined. The infinite-momentum/zero-mass limit

corresponds to the large-radius limit. Indeed, this limiting process in our case is to let the boost parameter  $\eta$  go to infinity, renormalize the larger components and let the smaller components go to zero.

In Sec. III, we discussed the contraction of the rotation generators in the  $SL(2, C)$  framework. In this section, we shall study the infinite-momentum/zero-mass limits of the spinorial combinations discussed in the preceding section. This problem has been studied for four-vectors in the literature. In this paper, we shall use the two-by-two representation of the four-vector to study this problem. As for the tensors, we are dealing with a new problem in this paper. We shall demonstrate in this section the tensor in (4.4) becomes the Maxwell field tensor for massless particles with spin 1.

### A. Four-Potential

For the case of a massless particle let us consider a photon moving in the  $z$  direction. The four-momentum vector is then  $p^\mu = (0, 0, p_3, p_0)$ . The mass-shell requirement  $p^\mu p_\mu = 0$ , together with the Lorentz condition

$$\partial_\mu A^\mu = P_\mu A^\mu = 0. \quad (5.1)$$

yields

$$A_t = A_z. \quad (5.2)$$

In the infinite- $\eta$  limit the identifications (4.2) represent the massless case and therefore are rewritten as the components of the four-potential

$$A_x \simeq \frac{1}{2}(\alpha\dot{\alpha} + \beta\dot{\beta}), \quad A_y \simeq \frac{i}{2}(\alpha\dot{\alpha} - \beta\dot{\beta}), \quad A_z = A_t \simeq \frac{1}{2}\alpha\dot{\beta}. \quad (5.3)$$

In the light cone coordinate system the two-by-two matrix for the potential takes the form

$$A = \begin{pmatrix} A_u & A_x - iA_y \\ A_x + iA_y & 0 \end{pmatrix}, \quad (5.4)$$

where  $A_u = (A_t + A_z)/\sqrt{2}$  and  $A_v = (A_t - A_z)/\sqrt{2} = 0$ . The vanishing of  $A_v$  in the infinite- $\eta$  limit is in accordance with the gauge condition (5.1). The transformation properties of the four-vector potential under the action of the little group represented by a four-by-four matrix is readily available in the literature [9]. Now, we have a two-by-two matrix for the four-potential, and the  $E(2)$ -like little group acts on  $A$  as

$$A' = D(u, v) A D(u, v)^\dagger, \quad (5.5)$$

where  $D(u, v)$  is given in (3.12). Then

$$A' = \begin{pmatrix} A_u + uA_x + vA_y & A_x - iA_y \\ A_x + iA_y & 0 \end{pmatrix}. \quad (5.6)$$

We noted that, for massless particles, the spinors of (3.13) are gauge-invariant, while those of (3.14) are gauge-dependent. The gauge-dependence of the four-potential comes from those gauge-dependent spinors. This is also why the four-potential is not directly observable. Is it then possible to construct state vectors solely from gauge-invariant spinors. This question was first addressed by Weinberg [10]. As we shall see in the following section, the gauge-invariant Maxwell field tensor is composed of those state vectors.

### B. Maxwell Field Tensor

If we follow the contraction procedure, the  $f'$  and  $g'$  components become the  $E$  and  $B$  components, where

$$\begin{aligned} E_x &\simeq \frac{1}{4}(\alpha\alpha + \dot{\beta}\dot{\beta}), & B_x &\simeq \frac{i}{4}(-\alpha\alpha + \dot{\beta}\dot{\beta}) \\ E_y &\simeq \frac{i}{4}(\alpha\alpha - \dot{\beta}\dot{\beta}), & B_y &\simeq \frac{1}{4}(\alpha\alpha + \dot{\beta}\dot{\beta}) \end{aligned} \quad (5.7)$$

and hence  $E_x = B_y$  and  $E_y = -B_x$ , which manifest the properties of electromagnetic fields for a circularly polarized photon propagating in the  $z$  direction.

We can now write the tensor as

$$F = \begin{pmatrix} 0 & 0 & B_y & E_x \\ 0 & 0 & -B_x & E_y \\ -B_y & B_x & 0 & 0 \\ -E_x & -E_y & 0 & 0 \end{pmatrix}. \quad (5.8)$$

This expression is only for the field propagating along the  $z$  direction. In general, this tensor can be rotated to suit an arbitrary direction. Then, the tensor takes the familiar form

$$F = \begin{pmatrix} 0 & -B_z & B_y & E_x \\ B_z & 0 & -B_x & E_y \\ -B_y & B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix}. \quad (5.9)$$

It is important to note that the second-rank tensor  $T$  in (4.4) is quite different from the above expression. The tensor  $T$  is for a massive particle and forms a representation space for the  $O(3)$ -like little group.

From the  $SL(2, C)$  stand point, there is a crucial difference between the four-potential and the electromagnetic field. We have seen in the preceding section that the potential is represented by the direct products of one dotted and one undotted spinor. On the other hand, electromagnetic fields are only represented by the gauge-invariant spinors of (3.13), where the direct product is between spinors which are either both dotted or both undotted. Furthermore, unlike the transformations on the four-potentials which can be made unitary [6], the transformations on the electromagnetic fields are nonunitary.

## CONCLUDING REMARKS

We are not the first to construct all possible representations using two  $SL(2, C)$  spinors. In this paper, we considered them in terms of Wigner's concept of little groups for internal space time symmetries. We considered them also in terms of the Inönü-Wigner contraction which leads to the idea of the  $E(2)$ -like little group as an infinite-momentum/zero-mass limit of the  $O(3)$ -like little group. We now have achieved a full understanding of the Maxwell fields and potentials in terms of Winger's little group.

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## APPENDIX A: TWO-BY-TWO MATRIX REPRESENTATION OF THE FOUR-VECTOR

The two-by-two matrix representation  $V$  for a four-vector as given in (4.1) is well known. It simply works! In this appendix, we would like to show why it works. The rotation subgroup of  $SL(2, C)$  is well known, and is called  $SU(2)$ . Then rotation on this matrix is achieved by

$$V' = R V R^\dagger. \quad (A1)$$

The rotation matrices are Hermitian, and its Hermitian conjugate  $R^\dagger$  is its inverse.

On the other hand, the story is quite different for boost matrices. Consider the boost matrix  $B_3$  along the  $z$  direction as given in (3.7). The boost along an arbitrary direction takes the form

$$B(\eta) = S B_3(\eta) S^\dagger, \quad (A2)$$

where  $S$  is a rotation matrix. Unlike the rotation matrix, the boost is not represented by a Hermitian matrix. The inverse of this matrix is not its Hermitian conjugate, but it is its “dot-conjugate”:

$$\dot{B}(\eta) = S B_3(-\eta) S^\dagger. \quad (A3)$$

Is it then possible to obtain this form by a matrix conjugation? Let us consider the matrix



$$g = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A4})$$

This matrix commutes with  $\sigma_2$  but anticommutes with  $\sigma_1$  and  $\sigma_3$ . For this reason,

$$\sigma_i = -g\sigma_i^T g^{-1}, \quad (\text{A5})$$

where  $\sigma_i^T$  is the transpose of  $\sigma_i$ . Let us call the above operation the “g-conjugation”. The g-conjugation of the rotation matrix results in its Hermitian conjugate:

$$R^{-1} = R^\dagger = gR^T g^{-1}, \quad S^{-1} = S^\dagger = gS^T g^{-1}. \quad (\text{A6})$$

The g-conjugation of the boost matrix results in the inverse or its dot-conjugation:

$$\dot{B} = B^{-1} = gB^T g^{-1}. \quad (\text{A7})$$

Next, what does the matrix  $g$  do on the spinors. It converts  $\alpha$  into  $\beta$ , and  $\beta$  into  $-\alpha$ . It converts also  $\dot{\alpha}$  into  $\dot{\beta}$ , and  $\dot{\beta}$  into  $-\dot{\alpha}$ . With this point in mind, we can consider the matrix

$$X = \begin{pmatrix} \alpha\dot{\beta} & -\alpha\dot{\alpha} \\ \beta\dot{\beta} & -\beta\dot{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \dot{\beta} & -\dot{\alpha} \end{pmatrix}. \quad (\text{A8})$$

The rotation of this matrix is done by

$$X' = RXR^{-1} = RXR^\dagger, \quad (\text{A9})$$

while it is boosted by the formula

$$X' = BXB = BXB^\dagger. \quad (\text{A10})$$

Furthermore, the components of the above  $X$  matrix can be identified with their respective counterparts in the matrix  $V$  given in (4.1).

## APPENDIX B: COMPARISON OF THE PRESENT NOTATION WITH THE CONVENTIONAL NOTATION

The purpose of this appendix is to clarify the relation between the present choice of the representation space with the one that frequently appears in literature [2]. The dotted and the undotted spinors are denoted by  $\xi^a = (\xi^1, \xi^2)$  and  $\eta^{\dot{a}} = (\eta^{\dot{1}}, \eta^{\dot{2}})$ , where the action of any element of the group  $SL(2, C)$  transforms the undotted spinors as:

$$\xi'^1 = a\xi^1 + c\xi^2, \quad \xi'^2 = b\xi^1 + d\xi^2. \quad (\text{B1})$$

Here  $a, b, c, d$  are complex functions of the six parameters of the group and are subject to  $ad - bc = 1$ . As to the transformation properties of the dotted spinors one has to introduce the complex conjugate transformation:

$$\eta'^{\dot{1}} = a^*\eta^{\dot{1}} + c^*\eta^{\dot{2}}, \quad \eta'^{\dot{2}} = b^*\eta^{\dot{1}} + d^*\eta^{\dot{2}}. \quad (\text{B2})$$

This is possible if one sets  $\xi^1 = (1, 0)^T$ ,  $\xi^2 = (0, 1)^T$  but  $\eta^{\dot{1}} = (0, 1)^T$ ,  $\eta^{\dot{2}} = (1, 0)^T$ . By comparison we thus have  $\xi^a = (\alpha, \beta)$  and  $\eta^{\dot{a}} = (\dot{\beta}, -\dot{\alpha})$ . Since choosing the above representation space admits only one of the solutions (i.e., solution (2.3)) of the commutation relations of the Lie algebra of  $SL(2, C)$ , the representation space as given in (3.4) and (3.5) is actually a natural consequence of having two nonequivalent solutions to the commutation relations of the algebra rather than choosing a convention.

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